

STATUS OF NNLO 3-JET CALCULATIONS

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The process $e^+e^- \rightarrow 3$ jets offers the opportunity to measure the strong coupling constant. For an accurate determination, precise theoretical calculations are necessary. I will give an overview on the status of the next-to-next-to-leading order calculations.

1 Introduction

Today's and future precision measurements in particle physics require accurate theoretical predictions in order to extract the values of the fundamental parameters of the theory from experiment. The precise values of these constants also serve to constrain the parameter space of models of new physics. To this aim existing next-to-leading order (NLO) predictions have to be extended to next-to-next-to-leading order (NNLO). While for inclusive observables like the total hadronic cross section in e^+e^- annihilation this step has been taken long time ago, the situation is much more complicated for exclusive quantities like event shapes or jet rates. One prominent process where a NNLO calculation for fully exclusive observables is desirable is $e^+e^- \rightarrow 3$ jets. The strong coupling constant α_s can be measured by using the data for $e^+e^- \rightarrow 3$ jets. A NNLO calculation is expected to reduce significantly the theoretical uncertainty in the extraction of α_s . A program for a fully exclusive NNLO calculation is flexible and allows to take into account complicated detector geometries and jet definitions. The observable may even be defined after the program has been written. The only requirement on the observable is infrared-safety. To construct an NNLO program the following ingredients are needed:

- the relevant scattering amplitudes up to two loops;
- a method to cancel infrared divergences;

There has been significant progress in the past years for each of these items and I will review them in the remainder of this talk.

2 The Calculation Of Two-Loop Amplitudes

The NNLO calculation of $e^+e^- \rightarrow 3$ jets requires the following amplitudes: the Born amplitudes for $e^+e^- \rightarrow 5$ partons, the one-loop amplitudes for $e^+e^- \rightarrow 4$ partons and the two-loop amplitudes for $e^+e^- \rightarrow 3$ partons. The necessary

Born amplitudes^{1,2} and the one-loop amplitudes^{3,4} have been obtained already a while ago. The most complicated parts are the two-loop amplitudes. This is due to new types of Feynman integrals, corresponding to Feynman diagrams with two loops and an external off-shell leg. Methods invented to tackle these integrals comprise the use of the Mellin-Barnes formula⁵, the application of differential equations and integration-by-parts identities⁶ as well as the use of nested sums⁷. The results are expressed in terms of (multiple) polylogarithms, thus extending the well known set of basic functions for one-loop integrals (logarithm and dilogarithm) and making contact with recent developments in mathematics. In addition to the results for the basic integrals, various reduction algorithms are also used: An algorithm by Tarasov⁸ allows to convert integrals with the loop momentum in the numerator into scalar integrals with possibly raised powers of the propagators and shifted dimension. An efficient algorithm by Laporta⁹ expresses a large set of unknown integrals in terms of a few “master” integrals. With these tools the two-loop amplitudes for $e^+e^- \rightarrow qg\bar{q}$ have been calculated^{10,11}. The two-loop amplitude has the colour decomposition

$$\begin{aligned} \mathcal{A}_3^{(2)} = & N^2 \mathcal{A}_{3,1}^{(2)} + \mathcal{A}_{3,2}^{(2)} + \frac{1}{N^2} \mathcal{A}_{3,3}^{(2)} + N_f N \mathcal{A}_{3,4}^{(2)} + \frac{N_f}{N} \mathcal{A}_{3,5}^{(2)} + N_f^2 \mathcal{A}_{3,6}^{(2)} \\ & + \mathcal{A}_{3,vec}^{(2)} + \mathcal{A}_{3,ax}^{(2)}. \end{aligned}$$

$\mathcal{A}_{3,vec}^{(2)}$ and $\mathcal{A}_{3,ax}^{(2)}$ are the contributions resulting from the vector and axial vector coupling of the photon/Z-boson to a closed quark loop. The Durham group calculated the partial amplitudes $\mathcal{A}_{3,1}^{(2)}$ - $\mathcal{A}_{3,6}^{(2)}$ as well as $\mathcal{A}_{3,vec}^{(2)}$. Our group obtained results for $\mathcal{A}_{3,4}^{(2)}$ - $\mathcal{A}_{3,6}^{(2)}$. The results from the two groups for $\mathcal{A}_{3,4}^{(2)}$ - $\mathcal{A}_{3,6}^{(2)}$ agree analytically. The partial amplitude $\mathcal{A}_{3,ax}^{(2)}$ remains to be calculated, although it is expected that this partial amplitude gives a negligible numerical contribution.

3 Cancellation Of Infrared Divergences

At the next-to-next-to-leading order level the ingredients for the third order term in the perturbative expansion for quantities depending on n resolved “hard” partons are the already mentioned n -parton two-loop amplitudes, the $(n+1)$ -parton one-loop amplitudes and the $(n+2)$ Born amplitudes. Taken separately, each one of these contributions is infrared divergent. Only the sum of all contributions is infrared finite. Infrared divergences occur already at next-to-leading order. At NLO real and virtual corrections contribute. The

virtual corrections contain the loop integrals and can have, in addition to ultraviolet divergences, infrared divergences. If loop amplitudes are calculated in dimensional regularisation, the IR divergences manifest themselves as explicit poles in the dimensional regularisation parameter $\varepsilon = 2 - D/2$. These poles cancel with similar poles arising from amplitudes with additional partons but less internal loops, when integrated over phase space regions where two (or more) partons become “close” to each other. In general, the Kinoshita-Lee-Nauenberg theorem^{12,13} guarantees that any infrared-safe observable, when summed over all states degenerate according to some resolution criteria, will be finite. However, the cancellation occurs only after the integration over the unresolved phase space has been performed and prevents thus a naive Monte Carlo approach for a fully exclusive calculation. It is therefore necessary to cancel first analytically all infrared divergences and to use Monte Carlo methods only after this step has been performed. At NLO, general methods to circumvent this problem are known. This is possible due to the universality of the singular behaviour of the amplitudes in soft and collinear limits. Examples are the phase-space slicing method and the subtraction method. Within the subtraction method^{14,15} one subtracts a suitable approximation term $d\sigma^A$ from the real corrections $d\sigma^R$. This approximation term must have the same singularity structure as the real corrections. If in addition the approximation term is simple enough, such that it can be integrated analytically over a one-parton subspace, then the result can be added back to the virtual corrections $d\sigma^V$. Since by definition $d\sigma^A$ has the same singular behaviour as $d\sigma^R$, the combination $(d\sigma^R - d\sigma^A)$ is integrable and can be evaluated numerically. Secondly, the analytic integration of $d\sigma^A$ over the one-parton subspace will yield the explicit poles in ε needed to cancel the corresponding poles in $d\sigma^V$. At NNLO this generalizes as follows¹⁶:

$$\begin{aligned}
\langle \mathcal{O} \rangle_n^{NNLO} = & \int \mathcal{O}_{n+2} d\sigma_{n+2}^{(0)} - \mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)} - \mathcal{O}_n \circ d\alpha_n^{(0,2)} \\
& + \int \mathcal{O}_{n+1} d\sigma_{n+1}^{(1)} + \mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)} - \mathcal{O}_n \circ d\alpha_n^{(1,1)} \\
& + \int \mathcal{O}_n d\sigma_n^{(2)} + \mathcal{O}_n \circ d\alpha_n^{(0,2)} + \mathcal{O}_n \circ d\alpha_n^{(1,1)}.
\end{aligned}$$

Here $d\alpha_{n+1}^{(0,1)}$ is a subtraction term for single unresolved configurations of Born amplitudes. This term is already known from NLO calculations. The term $d\alpha_n^{(0,2)}$ is a subtraction term for double unresolved configurations. Finally, $d\alpha_n^{(1,1)}$ is a subtraction term for single unresolved configurations involving one-loop amplitudes.

To construct these terms the universal factorisation properties of QCD amplitudes in unresolved limits are essential. QCD amplitudes factorise if they are decomposed into primitive amplitudes. Primitive amplitudes are defined by a fixed cyclic ordering of the QCD partons, a definite routing of the external fermion lines through the diagram and the particle content circulating in the loop. One-loop amplitudes factorise in single unresolved limits as^{17,18,19,20}

$$A_n^{(1)} = \text{Sing}^{(0,1)} \cdot A_{n-1}^{(1)} + \text{Sing}^{(1,1)} \cdot A_{n-1}^{(0)}.$$

Tree amplitudes factorise in the double unresolved limits as^{21,22,23,24,25}

$$A_n^{(0)} = \text{Sing}^{(0,2)} \cdot A_{n-2}^{(0)}.$$

The subtraction terms can be derived by working in the axial gauge. In this gauge only diagrams where the emission occurs from external lines are relevant for the subtraction terms. Alternatively, they can be obtained from off-shell currents and antenna factorisation²⁶.

4 Outlook

In this talk I reviewed the status of NNLO 3-jet calculations. With the progress we witnessed in the field in the last years we can expect to obtain numerical results rather soon and to extend existing numerical programs for NLO predictions on $e^+e^- \rightarrow 4$ jets^{27,28,29,30} towards NNLO predictions for $e^+e^- \rightarrow 3$ jets. In fact, a result for a particular colour structure for the thrust observable was announced recently³¹.

1. F. A. Berends, W. T. Giele, and H. Kuijf, Nucl. Phys. **B321**, 39 (1989).
2. K. Hagiwara and D. Zeppenfeld, Nucl. Phys. **B313**, 560 (1989).
3. Z. Bern, L. Dixon, D. A. Kosower, and S. Weinzierl, Nucl. Phys. **B489**, 3 (1997); Z. Bern, L. Dixon, and D. A. Kosower, Nucl. Phys. **B513**, 3 (1998).
4. E. W. N. Glover and D. J. Miller, Phys. Lett. **B396**, 257 (1997); J. M. Campbell, E. W. N. Glover, and D. J. Miller, Phys. Lett. **B409**, 503 (1997).
5. V. A. Smirnov, Phys. Lett. **B491**, 130 (2000); V. A. Smirnov, Phys. Lett. **B500**, 330 (2001).
6. T. Gehrmann and E. Remiddi, Nucl. Phys. **B580**, 485 (2000); Nucl. Phys. **B601**, 248 (2001); Nucl. Phys. **B601**, 287 (2001).
7. S. Moch, P. Uwer, and S. Weinzierl, J. Math. Phys. **43**, 3363 (2002); S. Weinzierl, Comput. Phys. Commun. **145**, 357 (2002).

8. O. V. Tarasov, Phys. Rev. **D54**, 6479 (1996); Nucl. Phys. **B502**, 455 (1997).
9. S. Laporta, Phys. Lett. **B504**, 188 (2001); Int. J. Mod. Phys. **A15**, 5087 (2000).
10. L. W. Garland, T. Gehrmann, E. W. N. Glover, A. Koukoutsakis, and E. Remiddi, Nucl. Phys. **B627**, 107 (2002); Nucl. Phys. **B642**, 227 (2002);
11. S. Moch, P. Uwer, and S. Weinzierl, Phys. Rev. **D66**, 114001 (2002).
12. T. Kinoshita, J. Math. Phys. **3**, 650 (1962).
13. T. D. Lee and M. Nauenberg, Phys. Rev. **133**, B1549 (1964).
14. S. Frixione, Z. Kunszt, and A. Signer, Nucl. Phys. **B467**, 399 (1996).
15. S. Catani and M. H. Seymour, Nucl. Phys. **B485**, 291 (1997); S. Dittmaier, Nucl. Phys. **B565**, 69 (2000); L. Phaf and S. Weinzierl, JHEP **04**, 006 (2001); S. Catani, S. Dittmaier, M. H. Seymour, and Z. Trocsanyi, Nucl. Phys. **B627**, 189 (2002).
16. S. Weinzierl, JHEP **03**, 062 (2003); JHEP **07**, 052 (2003).
17. Z. Bern, L. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B425**, 217 (1994).
18. Z. Bern, V. Del Duca, and C. R. Schmidt, Phys. Lett. **B445**, 168 (1998); Z. Bern, V. Del Duca, W. B. Kilgore, and C. R. Schmidt, Phys. Rev. **D60**, 116001 (1999).
19. D. A. Kosower, Nucl. Phys. **B552**, 319 (1999); D. A. Kosower and P. Uwer, Nucl. Phys. **B563**, 477 (1999).
20. S. Catani and M. Grazzini, Nucl. Phys. **B591**, 435 (2000).
21. F. A. Berends and W. T. Giele, Nucl. Phys. **B313**, 595 (1989).
22. A. Gehrmann-De Ridder and E. W. N. Glover, Nucl. Phys. **B517**, 269 (1998).
23. J. M. Campbell and E. W. N. Glover, Nucl. Phys. **B527**, 264 (1998).
24. S. Catani and M. Grazzini, Phys. Lett. **B446**, 143 (1999); Nucl. Phys. **B570**, 287 (2000).
25. V. Del Duca, A. Frizzo, and F. Maltoni, Nucl. Phys. **B568**, 211 (2000).
26. D. A. Kosower, Phys. Rev. **D57**, 5410 (1998); Phys. Rev. **D67**, 116003 (2003); Phys. Rev. Lett. **91**, 061602 (2003).
27. L. Dixon and A. Signer, Phys. Rev. **D56**, 4031 (1997).
28. Z. Nagy and Z. Trocsanyi, Phys. Rev. **D59**, 014020 (1999).
29. J. M. Campbell, M. A. Cullen, and E. W. N. Glover, Eur. Phys. J. **C9**, 245 (1999).
30. S. Weinzierl and D. A. Kosower, Phys. Rev. **D60**, 054028 (1999).
31. A. Gehrmann-De Ridder, T. Gehrmann, and E. W. N. Glover, hep-ph/0407023.